Parameter Estimation of Chirp Signals

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Abstract—**The problem of the parameter estimation of chirp signals is addressed. Several closely related estimators are proposed whose main characteristics are simplicity, accuracy, and ease of on-line or off-line implementation. For moderately high signal-to-noise ratios they are unbiased and attain the Cramer–Rao bound. Monte Carlo simulations verify the expected performance of the estimators.**

I. Introduction

CHIRP signals are common in various areas of science and engineering (e.g., physics, sonar, radar, communications). For example, they are used to estimate trajectories of moving objects with respect to fixed receivers. In addition, in situations where interference rejection is important, chirp signals provide a successful digital modulation scheme. The estimation of their parameters has been of interest for a long time. Most of the methods that have been suggested in the literature yield maximum likelihood estimates. A necessary condition for the derivation of these estimators is a strong signal-to-noise ratio. In [2] the method of inverse probability was used, while in [6] the estimation problem reduced to a location of the maximum of a multidimensional Gaussian function. The search for this maximum may be performed by Newton's method cited in [1]. In [7] the principles of linear least squares analysis were invoked. It was assumed that frequency or phase data were available on which a linear regression was performed. A similar method was proposed in [13] for estimating the frequency of a noisy sinusoid. The theoretical evaluation of the estimator presented there provided a major insight for better understanding the type of problem under consideration. Finally, in [8] rank reduction techniques were applied to estimate the parameters. The degree of the polynomial in the exponent of the chirp was successively transformed in order to reduce the chirp to a sinusoid. Then rank reduction was applied, followed by a dechirping of the original sequence and estimation of sinusoidal frequency.

This paper also addresses the problem of the parameter estimation of chirp signals. The derivation and schemes of several but related estimators will be given. The focus will be on jointly estimating the parameters that appear in the exponent of the chirp signal. Basically, the phase of the observed sequence is modeled as a polynomial embedded in white noise, which implies that, first, a phase unwrapping is accomplished, and, second, linear regression techniques applied to obtain the estimates of the parameters. This approach is suitable, for example, for an instrumentation radar, such as a high precision tracker.

An estimator of the frequency rate will also be derived. It will be easy to extend this approach to signals having polynomials of any degree in the exponent. All the derivations will be done under the assumption that the signal-to-noise ratio is sufficiently high. In addition to being accurate, the estimators are very simple for off-line or online implementation.

The paper is organized as follows. First the problem is stated. Then a method for the joint estimation of phase, frequency, and frequency rate is proposed based on the application of least squares to the unwrapped phase of the signal. The unwrapping scheme is very simple and strongly exploits the phase nature of this class of signals. In the fourth section the estimation of the frequency rate is analyzed. A procedure is proposed which does not need phase unwrapping. The problem of ambiguity is discussed next. Difficulties that arise from $2\pi n$ uncertainty inherent in the phase observations are pointed out and a solution proposed. Some other relevant questions concerning the proposed estimators are also raised, such as the demodulation of frequency/phase modulated signals. Finally, simulation results are presented which confirm the expected excellent performance. The estimates are unbiased and achieve the Cramer–Rao bound for signal-to-noise ratios above 8 dB.

II. Statement of the Problem

A sequence $\{x_n\}_{n=n_0}^{n=n_0+N-1}$ is observed having the following format:

$$x_n = A \exp\left[j\left(2\pi\left(\frac{\alpha}{2}n^2 + fn\right) + \phi\right)\right] + \epsilon_n$$
$$n = n_0, n_0+1, \cdots, n_0+N-1 \quad (1)$$

where A is the amplitude of the signal and α, f, and ϕ are its frequency rate, initial frequency, and phase, respectively. $\{\epsilon_n\}_{n=n_0}^{n=n_0+N-1}$ is a segment of a zero mean complex Gaussian process [4] with variance σ_ϵ^2. The real and imaginary parts of ϵ_n are independent and white with equal variance. A will be considered a nuisance parameter and will not be estimated. If we know the sequence $\{x_n\}_{n=n_0}^{n=n_0+N-1}$ and the initial sample instant n_0, the problem will be to estimate the parameters α, f, and ϕ. We will assume that the time between successive samples is

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1. n will be taken dimensionless. Then f and α will also be dimensionless.

The function in the exponent in (1) is a polynomial of second degree. However, most of the discussion below will be valid for polynomials of any degree. Thus if

$$x_n = Ae^{jF(\boldsymbol{\theta}, n)} + \epsilon_n \qquad n = n_0, n_0 + 1, \cdots, n_0 + N - 1 \quad (2)$$

where

$$F(\boldsymbol{\theta}, n) = \theta_p n^p + \theta_{p-1} n^{p-1} + \cdots + \theta_1 n + \theta_0 \quad (3)$$

then, as for $p = 2$, we may want to estimate the vector $\boldsymbol{\theta}^T = [\theta_0 \;\; \theta_1 \;\; \cdots \;\; \theta_p]$. The polynomial (3) may be expressed in terms of $n - n_0$ or $n - n_0 - (N - 1)/2$ with a redefinition of the parameters θ_i. This will imply that the new initial sample instant is $n_0' = 0$ or $n_0' = -(N - 1)/2$ which may often be very convenient. However, we shall keep in our analysis the initial sample instant n_0 because it includes the two mentioned cases above.

III. Joint Estimation of Phase, Frequency, and Frequency Rate

As a first step in the joint estimation procedure, we shall use the main result from [13]. There Tretter showed that for a high enough signal-to-noise ratio, the sequence

$$x_n = A \exp\left[j(2\pi f n + \phi)\right] + \epsilon_n \qquad n = n_0, n_0 + 1, \cdots, n_0 + N - 1 \quad (4)$$

which represents a complex sinusoid in noise, may be approximated by

$$x_n \simeq A \exp\left[j(2\pi f n + \phi + w_n)\right] \qquad n = n_0, n_0 + 1, \cdots, n_0 + N - 1 \quad (5)$$

where $\{w_n\}$ is a phase noise sequence. Further, it can be shown that if ϵ_n is complex white Gaussian noise, then w_n is real white Gaussian noise, [3]. Tretter suggested the joint estimation of f and ϕ by linear regression on the observed signal phase since all the information needed for their estimation was contained there. In order to get the phase sequence, a phase unwrapping algorithm applied to the principal value of arg x_n had to be used. With the unwrapped phase available, it was a simple matter to estimate f and ϕ.

It can similarly be shown (see the Appendix), that the sequence given by (1) can be approximated by

$$x_n \simeq A \exp\left[j\left(2\pi\left(\frac{\alpha}{2} n^2 + fn\right) + \phi + w_n\right)\right] \qquad n = n_0, n_0 + 1, \cdots, n_0 + N - 1 \quad (6)$$

where the phase noise $\{w_n\}$ is real white and Gaussian [3]. It is obvious that we would be able to estimate, α, f, and ϕ as in the complex sinusoid case. A prerequisite for successful estimation as in the first case is to correctly accomplish the phase unwrapping. In the case of a complex sinusoid, whenever the angular frequency of the sinusoid is not near π or $-\pi$, it is not difficult to unwrap the phase. Here, however, the phase might change very much from sample to sample, which makes the phase unwrapping a critical step of the procedure. For example, consider the noiseless case of (1) when $\alpha = 0, f = 0.1$, $\phi = 1.0$ rad, and $n_0 = 10$. The phases of the first two samples are arg $x_{n_0} = 2.32\ \pi$ rad, and arg $x_{n_0+1} = 2.52\ \pi$ rad. If $\alpha = 0.1$ and the rest of the parameters are unchanged, arg $x_{n_0} = 12.32\ \pi$ rad and arg $x_{n_0+1} = 14.62\ \pi$ rad.

In Fig. 1 the block diagram shows a scheme by which the problem may be successfully solved. It exploits the nature of the phase being unwrapped. It also offers some new insights into the general problem of the parameter estimation of chirp signals. In building the scheme the following idea was used: before the inverse tangent routine is employed which is a many-to-one transformation, the original sequence must be transformed into one whose phase always falls in the interval $(-\pi, \pi)$. This transformation has to be invertible because we want to go back and restore the phase of the original sequence. Let the true phase sequence as given in (6) be

$$\Phi_n = \pi\alpha n^2 + 2\pi f n + \phi + w_n. \quad (7)$$

One way to assure invertibility is to use the finite difference operator Δ, defined by

$$\Delta\Phi_n = \Phi_n - \Phi_{n-1}. \quad (8)$$

If we use it twice on Φ_n, we get

$$\Delta^2\Phi_n = \Delta(\Delta\Phi_n) = \Delta(\Phi_n - \Phi_{n-1}) = \Phi_n - 2\Phi_{n-1} + \Phi_{n-2}. \quad (9)$$

After substituting the values of Φ_n from (7) into (9), we get

$$\Delta^2\Phi_n = 2\pi\alpha + \Delta^2 w_n. \quad (10)$$

If we restrict α to satisfy

$$-0.5 < \alpha < 0.5$$

and since $\{w_n\}$ is small (high signal-to-noise ratio), we expect that $\Delta^2\Phi_n$ should fall somewhere in the interval $(-\pi, \pi)$ which was our goal.

An easy way to carry out (9) is to evaluate

$$y_n = x_n x_{n-1}^* \qquad n = n_0 + 1, \cdots, n_0 + N - 1 \quad (11)$$

and

$$z_n = y_n y_{n-1}^* \qquad n = n_0 + 2, \cdots, n_0 + N - 1 \quad (12)$$

where * denotes a complex conjugate. The sequence $\{z_n\}$ has the phase described by (10). Now we may utilize the inverse tangent on $\{z_n\}$ and get the values $\{\Delta^2\Phi_n\}$. Since we want the phase $\{\Phi_n\}$ and not the double differenced phase, we perform an inverse transformation on $\{\Delta^2\Phi_n\}$. That is, we have to double integrate $\{\Delta^2\Phi_n\}$, or

$$I^2(\Delta^2\Phi_n) = I(I(\Delta^2\Phi_n)) \quad (13)$$

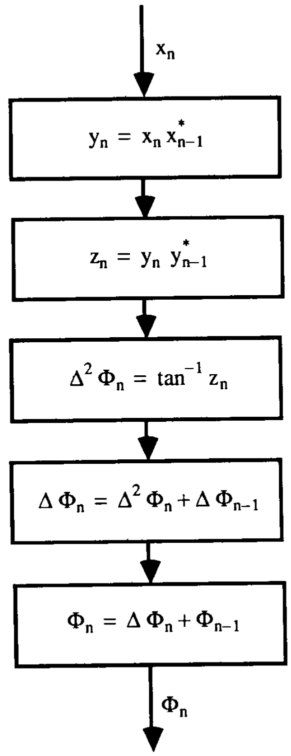

Fig. 1. Scheme I for unwrapping the phase of x_n (* denotes complex conjugation).

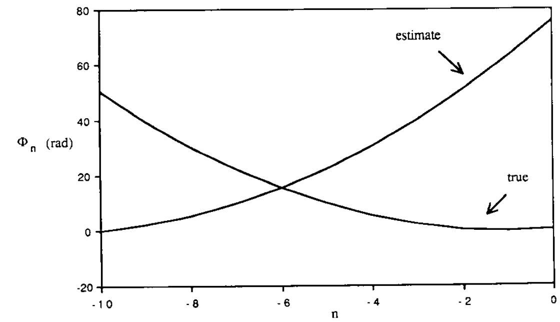

Fig. 2. Estimated and true phase sequence for a noiseless case ($n_0 = -10$, $N = 11$, $\alpha = 0.2$, $f = 0.2$, and $\phi = 0.2$ rad).

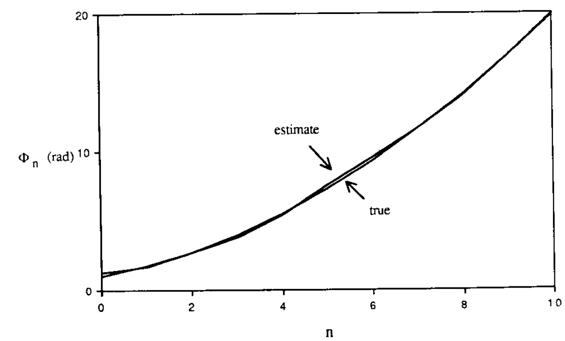

Fig. 3. Estimated and true phase sequence (SNR = 10 dB, $n_0 = 0$, $N = 11$, $\alpha = 0.04$, $f = 0.1$, and $\phi = 1.0$ rad).

where

$$I(\Delta^2 \Phi_n) = \Delta\Phi_n = \Delta^2\Phi_n + \Delta\Phi_{n-1}$$

and

$$I(\Delta\Phi_n) = \Phi_n = \Delta\Phi_n + \Phi_{n-1}.$$

In order to do (13) we need the initial samples of $\{\Delta\Phi_n\}$ and $\{\Phi_n\}$, i.e., $\Delta\Phi_{n0+1}$ and Φ_{n0}. Φ_{n0} will be the principal value of the first sample's argument, arg x_{n0}, and $\Delta\Phi_{n0+1}$ the principal value of the argument of $y_{n0+1} = x_{n0+1}x^*_{n0}$. Whenever $|\Phi_{n0}| > \pi$ and/or $|\Delta\Phi_{n0+1}| > \pi$, the estimated samples of the phase curve $\{\hat{\Phi}_n\}$ will differ from the true phase samples even when there is no noise. As an example in Fig. 2 the true and the estimated phase curves are shown for a noiseless case when $\alpha = 0.2$, $f = 0.2$, $\phi = 0.2$ rad, $n_0 = -10$, and $N = 11$. Although the curves look completely different, the estimated sequence still has the complete information about α, f, and ϕ since we assumed that we knew the initial sample instant n_0. Using this approach, we remove an important constraint that the classical routines for phase unwrapping have, i.e., that the frequency sampling must be fine enough that the difference between the phases of two consecutive samples is always less than a prescribed threshold. As an example, in Fig. 3 an unwrapped phase together with the true phase sequence is given for the case when the signal-to-noise ratio is 10 dB, $\alpha = 0.04$, $f = 0.1$, $\phi = 1.0$ rad, $n_0 = 0$, and $N = 11$. (The signal-to-noise ratio was defined as SNR = $10 \log_{10} A^2/\sigma_\epsilon^2$.)

Thus the procedure for estimating α, f, and ϕ would be the following:

a) Using the scheme in Fig. 1, unwrap the phase of $\{x_n\}_{n=n0}^{n=n0+N-1}$;

b) Evaluate $\hat{\theta}$ by

$$\hat{\theta} = (G^T G)^{-1} G^T \hat{\Phi} \tag{14}$$

where

$$\hat{\Phi}^T = [\hat{\Phi}_{n0} \;\; \hat{\Phi}_{n0+1} \;\; \cdots \;\; \hat{\Phi}_{n0+N-1}]$$

$$\hat{\Phi}_n = \Delta^2\hat{\Phi}_n + 2\hat{\Phi}_{n-1} - \hat{\Phi}_{n-2}$$

$$n = n_0 + 2, \cdots, n_0 + N - 1$$

$$\Delta^2\hat{\Phi}_n = \arg\,(x_n x^*_{n-1} x^*_{n-1} x_{n-2})$$

$$n = n_0 + 2, \cdots, n_0 + N - 1$$

$$\hat{\Phi}_{n0} = \arg x_{n0}$$

$$\hat{\Phi}_{n0+1} = \arg\,(x_{n0+1}x^*_{n0}) + \hat{\Phi}_{n0}$$

$$\hat{\theta}^T = [\hat{\phi}\,\hat{f}\,\hat{\alpha}]$$

and

$$G = \begin{bmatrix} 1 & n_0 & n_0^2 \\ 1 & n_0 + 1 & (n_0 + 1)^2 \\ \cdot & \cdot & \cdot \\ 1 & n_0 + N - 1 & (n_0 + N - 1)^2 \end{bmatrix}.$$

The estimates of α, f, and ϕ can be obtained sequentially by [10]:

$$\hat{\boldsymbol{\theta}}_{k+1} = \hat{\boldsymbol{\theta}}_k + \boldsymbol{P}_{k+1}\boldsymbol{g}_{k+1}(\hat{\Phi}_{k+1} - \boldsymbol{g}_{k+1}^T\hat{\boldsymbol{\theta}}_k) \quad (15)$$

and

$$\boldsymbol{P}_{k+1} = \boldsymbol{P}_k - \frac{\boldsymbol{P}_k\boldsymbol{g}_{k+1}\boldsymbol{g}_{k+1}^T\boldsymbol{P}_k}{(\boldsymbol{g}_{k+1}^T\boldsymbol{P}_k\boldsymbol{g}_{k+1} + 1)} \quad (16)$$

where $\hat{\boldsymbol{\theta}}_k$ is the estimate of $\boldsymbol{\theta}$ after k samples

$$\boldsymbol{g}_k^T = [1 \quad n_0 + k - 1 \quad (n_0 + k - 1)^2]$$

and

$$\boldsymbol{P}_k = [\boldsymbol{G}_k^T\boldsymbol{G}_k]^{-1}$$

where $\boldsymbol{G}_k$ is a submatrix of $\boldsymbol{G}$ with its k rows identical to the first k rows of $\boldsymbol{G}$.

IV. Frequency Rate Estimation

In practice it is often the case that the frequency rate is the only parameter of interest. In other words, having the sequence (1) we wish to estimate only α. Here an estimation procedure will be derived which is closely related to the approach described in the previous section.

We start again with the sequence given by (1). To estimate α we first form the sequence:

$$z_n = x_n x_{n-1}^* x_{n-1}^* x_{n-2} \qquad n = n_0 + 2, n_0 + 3, \cdots, n_0 + N - 1. \quad (17)$$

It is not hard to see that $\{z_n\}$ in (17) is identical to $\{z_n\}$ in (12). For a high signal-to-noise ratio it may be approximated (as (1) by (6); see the Appendix) as

$$z_n \simeq A^4 e^{j(2\pi\alpha + v_n)} \qquad n = n_0 + 2, n_0 + 3, \cdots, n_0 + N - 1 \quad (18)$$

where $\{v_n\}$ is a Gaussian noise process whose moments are

$$E\{\boldsymbol{v}\} = \boldsymbol{0} \quad (19)$$

$$E\{\boldsymbol{v}\boldsymbol{v}^T\} = \boldsymbol{C} = \frac{\sigma_\epsilon^2}{2A^2}\begin{bmatrix} 6 & -4 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -4 & 6 \end{bmatrix} \quad (20)$$

where $\boldsymbol{v}$ is an $(N-2) \times 1$ vector with components v_n, and $\boldsymbol{C}$ is $(N-2) \times (N-2)$ matrix. The information about α is contained in the phase of $\{z_n\}$, or

$$\psi_n = 2\pi\alpha + v_n \qquad n = n_0 + 2, n_0 + 3, \cdots, n_0 + N - 1. \quad (21)$$

Alternately,

$$\boldsymbol{\psi} = 2\pi\alpha\boldsymbol{1} + \boldsymbol{v} \quad (22)$$

where

$$\boldsymbol{1}^T = [1 \quad 1 \quad 1 \cdots 1]$$

is $1 \times (N-2)$. From (22) we obtain the Gauss–Markov estimator of α

$$\hat{\alpha} = \frac{1}{2\pi}\frac{\boldsymbol{1}^T\boldsymbol{C}^{-1}\boldsymbol{\psi}}{\boldsymbol{1}^T\boldsymbol{C}^{-1}\boldsymbol{1}} \quad (23)$$

where $\boldsymbol{C}$ is the covariance matrix of the phase noise sequence $\{v_n\}$. Note that $\{v_n\}$ is a real moving average process with driving noise variance $\sigma_\epsilon^2/2A^2$ and coefficients $b_0 = 1$, $b_1 = -2$, and $b_2 = 1$. It is obvious that for its implementation phase unwrapping is not needed since for $-0.5 < \alpha < 0.5$ the phase will be contained in the interval $(-\pi, \pi)$ at high signal-to-noise ratio. This implies that for this estimator we do not need to know n_0. To summarize, the estimation procedure consists of three steps:

1) transform $\{x_n\}$ into $\{z_n\}$ by (17);
2) evaluate the phase of $\{z_n\}$; and
3) use (23) to estimate α.

In order to further facilitate the use of this approach, we can show that the elements of the inverse matrix $\boldsymbol{C}^{-1}$ can be computed without actually inverting $\boldsymbol{C}$. With a procedure for inverting covariance matrices of ARMA processes presented in [11], the following expression for the elements of $\boldsymbol{C}^{-1}$ is obtained:

$$g_{i,j} = f_{i,j} - \frac{1}{f_{N-1,N-1}f_{N,N} - f_{N-1,N}^2}\{f_{i,N-1}f_{N,N}f_{j,N-1} + f_{i,N}f_{N-1,N-1}f_{j,N} - f_{N-1,N}(f_{i,N}f_{j,N-1} + f_{i,N-1}f_{j,N})\} \quad (24)$$

where the $f_{i,j}$'s are evaluated according to

$$f_{i,j} = \sum_{k=1}^{i} (j - i + k)k \qquad i \le j \le N.$$

For high signal-to-noise ratio the estimate of the frequency rate obtained using this approach is identical to the one obtained using the estimator described in the previous section because in both cases the estimate is obtained from the estimate of the double differenced phase sequence.

V. Discussion

If the values of α are close to 0.5 or -0.5, the presence of the phase noise samples $\{w_n\}$ in (7) may cause severe distortions in the phase unwrapping process. This will lead to useless estimates of α, f, and ϕ, even for fairly high signal-to-noise ratios which will not be acceptable.

To circumvent the problem, we shall further exploit the differencing of the signal phase. By adding an extra block to the scheme in Fig. 1 (see Fig. 4), we remove the constraint that for proper phase unwrapping the value of α should not be near ± 0.5. The role of the extra differencing can easily be deduced from (10). If $\Delta^2 \Phi_n$ takes values that are close to $-\pi$ or π, then the once more differenced values of the phase $\{\Delta^3 \Phi_n\}$ will be distributed around zero. If that is the case, the inverse tangent routine may be used safely. However, we should be aware that the more stages there are in the scheme, the higher the probability for an outlier occurrence. That is, each new stage introduces one more differencing of the phase noise. Since the phase noise sequence $\{w_n\}$ in (7) is stochastic, it is highly probable that a couple of successive samples may alternate in sign. If the samples of the several times differenced sequence become large enough to cross the boundaries of $-\pi$ or π, they will damage our estimates severely. If the variance of w_n in (7) is σ_w^2, then the variance of $\Delta^p w_n$ is

$$\text{var}(\Delta^p w_n) = \binom{2p}{p} \sigma_w^2.$$

Obviously, the noise distribution of $\Delta^p w_n$ will be broader for bigger values of p which will imply higher probability of outliers. Also, this probability grows when the length of the sequence to be unwrapped is longer. Thus, the more stages we use in the scheme, the higher the signal-to-noise ratio is required for proper performance. The computer simulations presented in the next section show that the scheme where the phase is differenced twice has a lower threshold by 4 dB than the scheme where the phase is differenced three times. However, when the magnitude of α is close to 0.5 or -0.5, the performance of this latter scheme is much better than the former.

The unwrapping scheme used in this paper is not applicable in general. It is not hard to show that if we use the difference operator k times, the necessary and sufficient conditions for correct phase unwrapping are as follows:

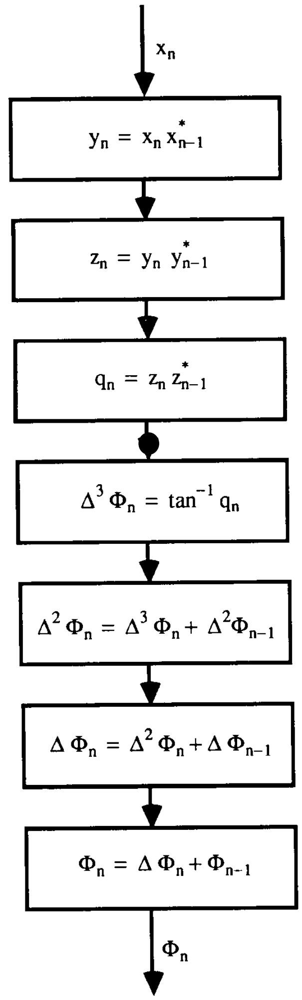

Fig. 4. Scheme II for unwrapping the phase of x_n (* denotes complex conjugation).

$$-\pi < \Delta^k \Phi_n < \pi$$

$$n = n_0 + k, n_0 + k + 1, \cdots, n_0 + N - 1 \qquad (25)$$

together with

$$-\pi < \Phi_{n0} < \pi$$
$$-\pi < \Delta\Phi_{n0+1} < \pi$$
$$-\pi < \Delta^2\Phi_{n0+2} < \pi$$
$$\cdots$$
$$-\pi < \Delta^{k-1}\Phi_{n0+k-1} < \pi$$

where $\Delta^k \Phi_{n0+k} = \Delta^{k-1}\Phi_{n0+k} - \Delta^{k-1}\Phi_{n0+k-1}$, $k = 1, 2, \cdots$ and $\Delta^0 \Phi_n = \Phi_n$.

It should also be noted that in order to estimate the parameters θ_i in (3) correctly using our method, they have to satisfy

$$|\theta_i| < \frac{\pi}{i!}.$$

Another attractive approach for the joint parameter estimates of chirp signals seems to be the following. When we have estimated the frequency rate, we might dechirp the sequence $\{x_n\}$, after which we are (hopefully) left with a complex sinusoid, $\{\tilde{x}_n\}$. In the next step we could use the method in [5] for frequency estimation. If we want the estimate of ϕ as well, we could further demodulate $\{\tilde{x}_n\}$ using the estimated frequency and get $\{\tilde{\tilde{x}}_n\}$. From the phase sequence of $\{\tilde{\tilde{x}}_n\}$ the phase estimate should be easily obtained. However, this procedure yields poor results (some of which will be presented) because we always dechirp or demodulate using estimated values of parameters, thus introducing in $\{\tilde{x}_n\}$ and $\{\tilde{\tilde{x}}_n\}$ drifts which further deteriorate the estimates of f and ϕ.

Tretter shows that his estimator achieves the Cramer–Rao bound for a moderately high signal-to-noise ratio [13]. Similarly, we will show that for our problem the proposed estimator also attains the Cramer–Rao bounds at high signal-to-noise ratio

$$\sigma_{\hat{\alpha}}^2 = \frac{\sigma_\epsilon^2}{A^2}\frac{1}{2\pi^2}\frac{QN - P^2}{SQN + 2RPQ - Q^3 - P^2S - R^2N} \tag{26}$$

$$\sigma_{\hat{f}}^2 = \frac{\sigma_\epsilon^2}{A^2}\frac{1}{8\pi^2}\frac{SN - Q^2}{SQN + 2RPQ - Q^3 - P^2S - R^2N} \tag{27}$$

and

$$\sigma_{\hat{\phi}}^2 = \frac{\sigma_\epsilon^2}{2A^2}\frac{SQ - R^2}{SQN + 2RPQ - Q^3 - P^2S - R^2N} \tag{28}$$

where

$$P = \sum_{n=n_0}^{n_0+N-1} n, \qquad Q = \sum_{n=n_0}^{n_0+N-1} n^2$$

$$R = \sum_{n=n_0}^{n_0+N-1} n^3, \qquad S = \sum_{n=n_0}^{n_0+N-1} n^4.$$

It is well known and obvious from these expressions that $\sigma_{\hat{\alpha}}^2 \sim (1/N^5)$, $\sigma_{\hat{f}}^2 \sim (1/N^3)$, and $\sigma_{\hat{\phi}}^2 \sim (1/N)$, and that the bounds depend on the initial sample instant. They attain the minimum when the sequences are centered around $n = 0$, i.e., $n_0 = -(N-1)/2$. In that case $P = R = 0$, and

$$\sigma_{\hat{\alpha}}^2 = \frac{\sigma_\epsilon^2}{A^2}\frac{1}{\pi^2}\frac{90}{N(N^2-1)(N^2-4)} \tag{29}$$

$$\sigma_{\hat{f}}^2 = \frac{\sigma_\epsilon^2}{A^2}\frac{1}{\pi^2}\frac{3}{2N(N^2-1)} \tag{30}$$

and

$$\sigma_{\hat{\phi}}^2 = \frac{\sigma_\epsilon^2}{A^2}\frac{9N^2 - 21}{8N(N^2-4)}. \tag{31}$$

Moreover, the curves that represent the Cramer–Rao bound of any of the general chirp signal parameters θ_i from (2) for fixed N are symmetric around $n_0 = -(N-1)/2$ when plotted versus the initial sample instant. While all these facts are more or less known, their physical explanation has not been quite obvious.

In our exposition of the problem we have assumed that the additive complex noise $\{\epsilon_n\}$ was white and Gaussian and the derivations were based on that assumption. The same estimators cannot be used when the noise is correlated because only the white Gaussian noise does not change its first- and second-order statistics after being modulated by a chirp signal (see the Appendix).

VI. Simulation Results

We have investigated the performance of the estimators for different signal-to-noise ratios and fixed parameters α, f, and ϕ and compared it with the Cramer–Rao bounds. The complex noise used in the simulation had a Gaussian distribution. On the y-axis $10 \log_{10} [1/(1/M)\Sigma_{k=1}^{M} (\theta - \hat{\theta}_k)^2]$ was plotted (where θ may be the frequency rate, the frequency, or the phase, $\hat{\theta}_k$ its estimate from the kth realization, M the number of realizations) and on the x axis the signal-to-noise ratio. There were 200 realizations per signal-to-noise ratio, starting from 0 to 30 dB in steps of 0.5 dB.

The estimates of ϕ, f, and α are presented in Figs. 5–7, respectively, together with the Cramer–Rao bounds. The sequences had $N = 31$ samples, with the initial sample instant being $n_0 = -15$. The true values of the parameters were $\phi = 1.0$ rad, $f = 0.3$, and $\alpha = 0.1$. In Fig. 5 there are two curves. One represents the performance of the estimator based on the scheme in Fig. 1, and the other the performance of the estimator based on the scheme in Fig. 4. As mentioned, the difference is evident for low signal-to-noise ratios. The more stages we use and the longer the sequences are, the more probable is the occurrence of an outlier, especially when the signal-to-noise ratio is relatively low. For example, when the ratio was 9 dB and the length of the sequences was 15, the simulations showed that in 5% of the cases there was an outlier. For the same signal-to-noise ratio and sequences having 31 samples, the figure increased to 10%. When the length of the sequences was further increased to $N = 61$, the figure rose to 12%. A similar difference in performance between these two estimators can be seen for the frequency and frequency rate (Figs. 6 and 7). In Fig. 6 there is one more curve; it shows the performance of the estimator which first estimates the frequency rate, then dechirps the original sequence and finally estimates the frequency of the dechirped signal. The procedure yielded poor results. The error in the estimate of the frequency rate introduces drift (or change) in the frequency of the

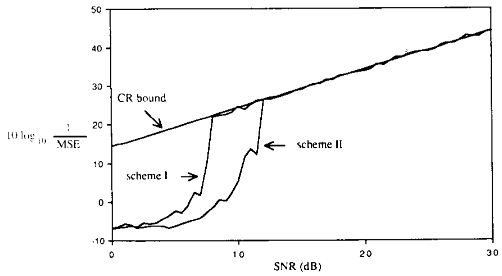

Fig. 5. Phase estimate performance ($n_0 = -15$, $N = 31$, $\alpha = 0.1$, $f = 0.3$, and $\phi = 1.0$ rad).

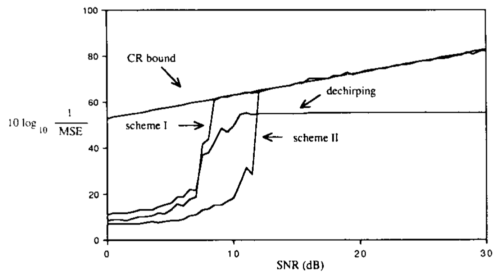

Fig. 6. Frequency estimate performance ($n_0 = -15$, $N = 31$, $\alpha = 0.1$, $f = 0.3$, and $\phi = 1.0$ rad).

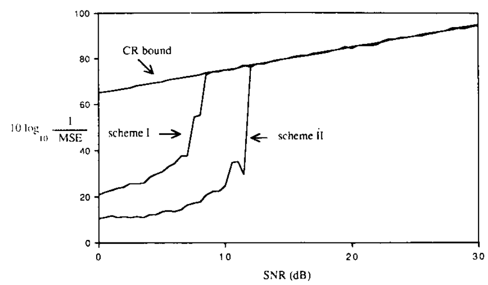

Fig. 7. Frequency rate estimate performance ($n_0 = -15$, $N = 31$, $\alpha = 0.1$, $f = 0.3$, and $\phi = 1.0$ rad).

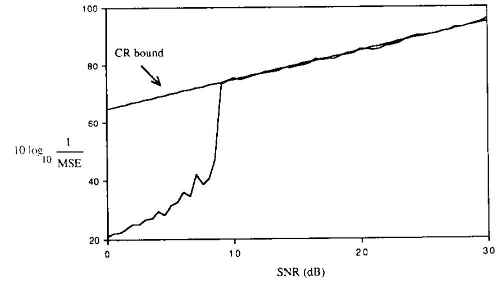

Fig. 8. Frequency rate estimate performance ($n_0 = 30$, $N = 31$, $\alpha = 0.1$, $f = 0.3$, and $\phi = 1.0$ rad).

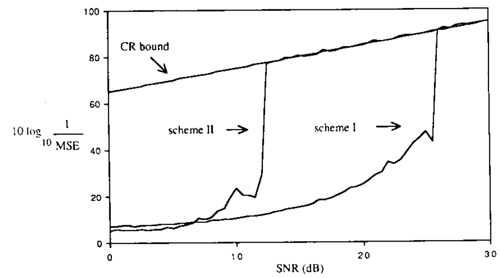

Fig. 9. Frequency rate estimate performance ($n_0 = -15$, $N = 31$, $\alpha = 0.45$, $f = 0.45$, and $\phi = 2.0$ rad).

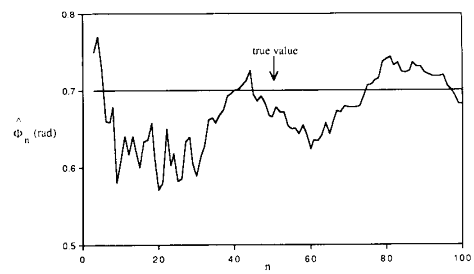

Fig. 10. Sequential phase estimates ($n_0 = 0$, $N = 100$, $\alpha = 0.1$, $f = 0.25$, $\phi = 0.7$ rad, and SNR = 10 dB).

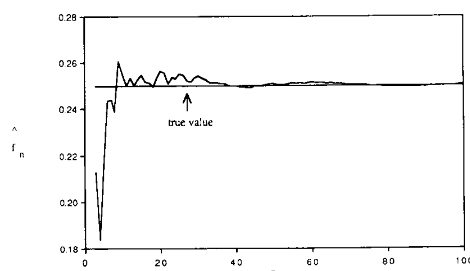

Fig. 11. Sequential frequency estimates ($n_0 = 0$, $N = 100$, $\alpha = 0.1$, $f = 0.25$, $\phi = 0.7$ rad, and SNR = 10 dB).

sinusoid that should be obtained after dechirping, which further strongly affects the estimates.

Similar figures can be obtained for different parameters of the observed sequence. In Fig. 8, the estimate of the frequency rate is given, together with the Cramer–Rao bound, for sequences with 31 samples. The parameters were the same as in the previous case, except that the initial sample instant was $n_0 = 30$. Note that in this sequence the phase changes much more rapidly from sample to sample than in the previous case.

In Fig. 9 the estimates of the frequency rate are shown for sequences with the following parameters: $N = 31$, $n_0 = -15$, $\alpha = 0.45$, $f = 0.45$, and $\phi = 2$ rad. One curve shows the performance of the estimator in Fig. 1, and the other of the estimator in Fig. 4. The estimator with 3 stages performs identically to the case when the value of α was 0.1, while the estimator with 2 stages performs poorly even for high signal-to-noise ratios because α is close to 0.5.

In [12] the threshold behavior of Tretter's frequency estimator was investigated and similar experimental re-

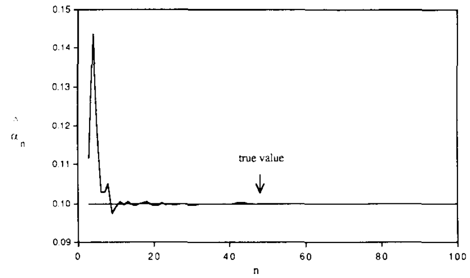

Fig. 12. Sequential frequency rate estimates ($n_0 = 0$, $N = 100$, $\alpha = 0.1$, $f = 0.25$, $\phi = 0.7$ rad, and SNR = 10 dB).

sults were reported as here. In [1] lower thresholds were reported, but at the cost of much more computation.

The estimators may be implemented to work on-line. In Figs. 10–12 typical sequential estimates of the phase, frequency and frequency rate are given. On the x axis the number of samples is plotted, and on the y axis the estimate at the particular sample. The straight lines show the true values of the parameters. The signal-to-noise ratio was 10 dB.

VII. Conclusion

In this paper the approaches we have proposed for joint estimation of frequency rate, frequency, and phase, and frequency rate alone are simple, accurate, and achieve the Cramer–Rao bound for signal-to-noise ratios higher than 8 dB. The same approach can be easily extended to cases in which the function in the exponent of the signal is a polynomial of any degree. However, increasing the degree also increases the threshold for proper performance. The estimators use the phase information only which makes their performance less susceptible to amplitude fluctuations in the data.

It will be interesting to investigate the applicability of these ideas for the digital demodulation of frequency or phase modulated signals and eventually compare the demodulators so designed with others, based on various approaches, for example, digital phase locked loop, Hilbert transform, etc. Most of the algorithms for digital demodulation require constant amplitude, or small frequency (or phase) deviation [9]. The approach here does not assume any of these. The only necessary condition for proper performance is a moderately high signal-to-noise ratio.

Appendix

We first want to show the validity of approximation (6). In doing so, we shall follow Tretter's derivation. Let $\{x_n\}_{n=n_0}^{n_0+N-1}$ be a finite sequence of samples described by

$$x_n = A\exp\left[j\left(2\pi\left(\frac{\alpha}{2}n^2 + fn\right) + \phi\right)\right] + \epsilon_n$$

$$n = n_0, n_0 + 1, \cdots, n_0 + N - 1$$

where $\{\epsilon_n\}$ is a sequence of complex white Gaussian noise whose real and imaginary parts are uncorrelated, each having variance $\sigma_\epsilon^2/2$. The samples x_n may be written as

$$x_n = A\exp\left[j\left(2\pi\left(\frac{\alpha}{2}n^2 + fn\right) + \phi\right)\right] \cdot \left(1 + \frac{\epsilon_n}{A}\exp\left[-j\left(2\pi\left(\frac{\alpha}{2}n^2 + fn\right) + \phi\right)\right]\right).$$

Let

$$u_n = \frac{\epsilon_n}{A}\exp\left[-j\left(2\pi\left(\frac{\alpha}{2}n^2 + fn\right) + \phi\right)\right].$$

In matrix notation

$$\boldsymbol{u} = \boldsymbol{\Gamma}\boldsymbol{\varepsilon}$$

where

$$\boldsymbol{u}^T = [u_{n_0} \quad u_{n_0+1} \quad \cdots \quad u_{n_0+N-1}]$$

$$\boldsymbol{\varepsilon}^T = [\epsilon_{n_0} \quad \epsilon_{n_0+1} \quad \cdots \quad \epsilon_{n_0+N-1}]$$

and

$$\boldsymbol{\Gamma} = \begin{bmatrix} \frac{1}{A}\exp\left[-j\left(2\pi\left(\frac{\alpha}{2}n_0^2 + fn_0\right) + \phi\right)\right] & 0 & 0 \cdot & 0 \\ 0 & \frac{1}{A}\exp\left[-j\left(2\pi\left(\frac{\alpha}{2}(n_0+1)^2 + f(n_0+1)\right) + \phi\right)\right] & 0 \cdot & 0 \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ 0 & 0 & \cdot\ \cdot & \frac{1}{A}\exp\left[-j\left(2\pi\ \frac{\alpha}{2}(n_0+N-1)^2 + f(n_0+N-1)\right) + \phi\right] \end{bmatrix}$$

Thus the original noise vector $\boldsymbol{\varepsilon}$ is linearly transformed by the matrix $\boldsymbol{\Gamma}$. It may be shown that the vector $\boldsymbol{u}$ also has a complex Gaussian distribution. If the covariance matrix of the original noise is σ_ϵ^2, $\boldsymbol{T}$, or is diagonal with nonequal elements, then the covariance matrix of the transformed noise remains unchanged. Thus, its real and imaginary parts represent real Gaussian vectors whose elements are uncorrelated with equal variance σ_ϵ^2/A^2.

Now $1 + u_n$ may be approximated for a high signal-to-noise ratio as

$$1 + u_n \simeq e^{ju_n^{(i)}}$$

where $u_n^{(i)}$ is the imaginary part of u_n, since

$$1 + u_n = 1 + u_n^{(r)} + ju_n^{(i)} = \left(\left(1 + u_n^{(r)}\right)^2 + \left(u_n^{(i)}\right)^2\right)^{1/2}\exp\left(j\arctan\frac{u_n^{(i)}}{1 + u_n^{(r)}}\right)$$

$u_n^{(r)}$ being the real part of u_n, and

$$1 + u_n^{(r)} \simeq 1$$

$$\left(\left(1 + u_n^{(r)}\right)^2 + \left(u_n^{(i)}\right)^2\right)^{1/2} \simeq 1.$$

Using these approximations, for the original sequence we obtain

$$x_n \simeq A \exp\left[j\left(2\pi\left(\frac{\alpha}{2}n^2 + fn\right) + \phi + u_n^{(i)}\right)\right]$$

which is (6), where for the imaginary part of u_n we used the notation w_n. Note that the variance of $\{w_n\}$ is approximately σ_ϵ^2/A^2.

Next, we want to show the validity of (18) and evaluate the second-order statistics of the noise process $\{v_n\}$. To do so, first form the sequence

$$z_n = x_n x_{n-1}^* x_{n-1}^* x_{n-2}$$
$$n = n_0 + 2, n_0 + 1, \cdots, n_0 + N - 1.$$

It can be shown that for a high signal-to-noise ratio $\{z_n\}$ may be approximated by

$$\begin{aligned} z_n \simeq A^4 e^{j2\pi\alpha}\Bigg(1 &+ \frac{\epsilon_n}{A} \cdot \exp\left[-j\left(2\pi\left(\frac{\alpha}{2}n^2 + fn\right) + \phi\right)\right] \\ &+ 2\frac{\epsilon_{n-1}^*}{A}\exp\left[j\left(2\pi\left(\frac{\alpha}{2}(n-1)^2 + f(n-1)\right) + \phi\right)\right] \\ &+ \frac{\epsilon_{n-2}}{A}\exp\left[-j\left(2\pi\left(\frac{\alpha}{2}(n-2)^2 + f(n-2)\right) + \phi\right)\right]\Bigg) \end{aligned}$$

after omitting terms of the form $(\epsilon_k \epsilon_{k-1}^*/A^2)$, $(\epsilon_k \epsilon_{k-1}^* \epsilon_{k-1}^*/A^3)$, $(\epsilon_k \epsilon_{k-1}^* \epsilon_{k-1}^* \epsilon_{k-2}/A^4)$, etc. Since

$$u_n = \frac{\epsilon_n}{A}\exp\left[-j\left(2\pi\left(\frac{\alpha}{2}n^2 + fn\right) + \phi\right)\right]$$

we can write

$$z_n \simeq A^4 e^{j2\pi\alpha}(1 + u_n + 2u_{n-1}^* + u_{n-2}).$$

Denoting the real and imaginary part of u_n as $u_n^{(r)}$ and $u_n^{(i)}$, respectively, we proceed similarly as before and finally get

$$\begin{aligned} z_n &\simeq A^4 e^{j2\pi\alpha}\left((1 + u_{n(z)} + 2u_{n-1}^{(r)} + u_{n-2}^{(r)})^2 + (u_n^{(i)} - 2u_{n-1}^{(i)} + u_{n-2}^{(i)})^2\right)^{1/2} \\ &\quad \cdot \exp\left[j \arctan\left(\frac{u_n^{(i)} - 2u_{n-1}^{(i)} + u_{n-2}^{(i)}}{1 + u_n^{(r)} + 2u_{n-1}^{(r)} + u_{n-2}^{(r)}}\right)\right] \\ &\simeq A^4 \exp\left[j(2\pi\alpha + u_n^{(i)} - 2u_{n-1}^{(i)} + u_{n-2}^{(i)})\right] \end{aligned}$$

which is the same as (18) if we denote

$$v_n = u_n^{(i)} - 2u_{n-1}^{(i)} + u_{n-2}^{(i)}.$$

Since $\{u_n^{(i)}\}$ is real white Gaussian, v_n is a moving average process with coefficients $b_0 = 1$, $b_1 = -2$, and $b_2 = 1$. Obviously $\{v_n\}$ is zero mean. The covariance matrix is now readily shown to be $\boldsymbol{C}$ in (20).

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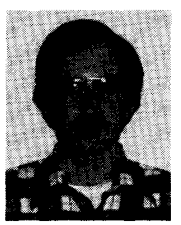

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