A RESOLUTION PERFORMANCE MEASURE FOR QUADRATIC TIME-FREQUENCY DISTRIBUTIONS

Boualem Boashash and Victor Sucic

Signal Processing Research Centre
Queensland University of Technology
GPO Box 2434, Brisbane, Qld 4001, Australia
e-mail: b.boashash@qut.edu.au, v.sucic@qut.edu.au

ABSTRACT

This paper presents two novel results which are significant for the application of time-frequency signal analysis techniques to real life signals. First, we introduce a measure for comparing the resolution performance of TFDs in separating closely spaced components in the time-frequency domain. The measure takes into account key attributes of TFDs such as main-lobes, side-lobes, and cross-terms. The introduction of this measure is an improvement of current techniques which rely on visual inspection of plots.

The second result consists in proposing a methodology for designing high resolution quadratic TFDs for the time-frequency analysis of multicomponent signals when components are close to each other. A recently introduced TFD, the B-distribution, and its modified version are defined using this methodology.

Finally, the performance comparison of quadratic TFDs using the proposed resolution measure shows that the B-distribution outperforms existing quadratic TFDs in resolving closely spaced components in the time-frequency domain.

1. INTRODUCTION

This paper describes what we believe is the first attempt at providing an objective quantitative measure criterion for comparing the performance of quadratic time-frequency distributions (TFDs), in terms of resolution (separation of closely spaced components), when applied to the analysis of multicomponent signals.

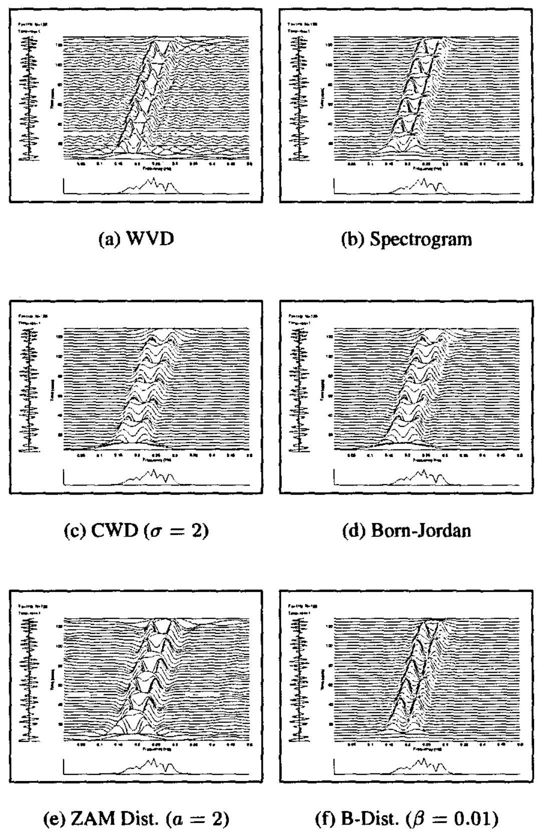
Let us consider a multicomponent signal given by:

$$s(t) = s_1(t) + s_2(t) \tag{1}$$

where $s_1(t)$ and $s_2(t)$ are two parallel linear frequency modulated (LFM) signals of length $N = 128$ and sampling frequency $f_s = 1Hz$. The frequency of the first component $s_1(t)$ goes from $0.15Hz$ to $0.25Hz$, while the frequency of the second component $s_2(t)$ varies from $0.2Hz$ to $0.3Hz$.

The multicomponent signal $s(t)$ is represented in the time-frequency domain using the Wigner-Ville distribution (WVD), the spectrogram, the Choi-Williams distribution (CWD) [1], the Born-Jordan distribution [2], Zhao-Atlas-Marks (ZAM) distribution [3], and the recently introduced B-distribution [4, 5] (see Figure 1).

The desire to objectively compare the plots in Figure 1 motivated the need to define a quantitative performance measure for TFDs. The characteristics of TFDs that influence their resolution, such as energy concentration, mainlobes separation, sidelobes and cross-terms minimisation, are combined to define a quantitative measure criterion.



(a) WVD (b) Spectrogram

(c) CWD ($\sigma = 2$) (d) Born-Jordan

(e) ZAM Dist. ($a = 2$) (f) B-Dist. ($\beta = 0.01$)

Figure 1: *TFDs of two LFMs with frequency $f_1 = 0.15 - 0.25Hz$ and $f_2 = 0.2 - 0.3Hz$. All plots use a rectangular window, apart from the spectrogram which uses the Hanning window*

This paper presents a comparison of the resolution performance of the above mentioned TFDs, using the newly proposed measure criterion. In this context, we show that the B-distribution outperforms the other quadratic TFDs for signals with components closely-spaced in the time-frequency plane.

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2. PERFORMANCE CRITERIA OF TIME-FREQUENCY DISTRIBUTIONS

2.1. Monocomponent Signal

The performance of a TFD in the case of *monocomponent* FM signals is commonly defined in terms of the energy concentration the TFD achieves about the signal instantaneous frequency (IF) [6]. For the slice of TFD taken at the time instant t_0, illustrated in Figure 2, we may express the performance measure as:

$$p = \frac{|A_S|}{|A_M|}\frac{V}{f} \tag{2}$$

where A_M is the amplitude of the mainlobe of the TFD, A_S is the amplitude of the sidelobes, V is the 1.5 dB bandwidth[1] of the mainlobe and f represents the IF of the signal, all taken at time t_0. The rationale for introducing (2) is that one wants to minimise sidelobe amplitude A_S and mainlobe bandwidth V relative to central frequency f, but maximise mainlobe amplitude A_M.

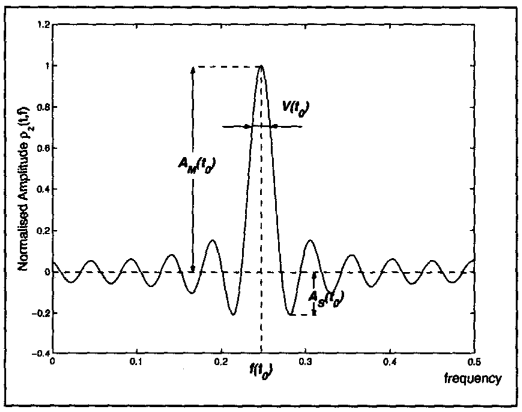


Figure 2: *Slice of a TFD of a monocomponent signal taken at the time instant* $t = t_0$

2.2. Multicomponent Signal

The performance of time-frequency distributions of a *multicomponent* FM signal, can be *quantitatively* measured in terms of:

- the energy concentration of the distribution about the respective instantaneous frequency of each component, as expressed by equation (2), and
- the resolution as measured by the separation of the mainlobes of the components in the time-frequency plane, and the effect of cross-terms.

2.2.1. *Energy Concentration*

By extending the concept in Section 2.1, a TFD is said to have the best energy concentration for a given multicomponent FM signal if for each of the signal components:

- its 1.5 dB mainlobe bandwidth relative to f is the smallest compared to that of other distributions, and if
- it yields the smallest sidelobe magnitude to mainlobe magnitude ratio compared to those of other distributions.

2.2.2. *Resolution*

The frequency resolution in a power spectral estimate of a signal composed of two single tones, f_1 and f_2, is defined as the minimum difference $f_2 - f_1$ for which the following inequality holds:

$$f_1 + \frac{V_1}{2} < f_2 - \frac{V_2}{2} \tag{3}$$

where V_1 and V_2 are the 1.5 dB mainlobe bandwidth of the first and the second sinusoid, respectively, as illustrated in Figure 3.

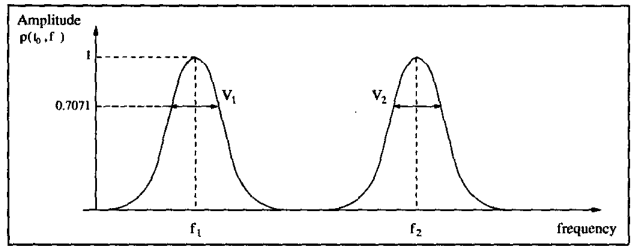


Figure 3: *Resolution of a two-component signal*

For a time-frequency distribution $\rho_z(t, f)$ of a two-component signal, the above definition of resolution would be valid for every slice of cross-terms free TFDs, such as the spectrogram, taken at time $t = t_0$. However, for TFDs with cross-terms, we need to account for the effect of cross-terms on resolution, as illustrated by Figure 4 and explained in the next section.

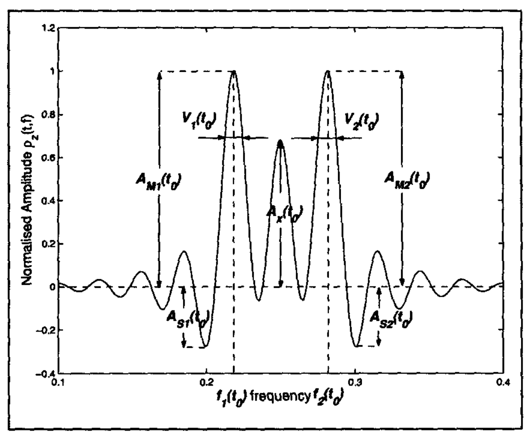


Figure 4: *Slice of a TFD of a two-component signal taken at time* $t = t_0$

In Figure 4, $V_1(t_0)$, $f_1(t_0)$, $A_{S_1}(t_0)$ and $A_{M_1}(t_0)$ represent respectively the 1.5 dB mainlobe bandwidth, the instantaneous frequency, the sidelobe amplitude and the mainlobe amplitude of the

[1] We measure the bandwidth of the mainlobe of a component at the rms value of the component normalised amplitude. See also footnote 5.

first component at time $t = t_0$. Similarly, $V_2(t_0)$, $f_2(t_0)$, $A_{S_2}(t_0)$ and $A_{M_2}(t_0)$ represent the 1.5 dB mainlobe bandwidth, the instantaneous frequency, the sidelobe amplitude and the mainlobe amplitude of the second component at the same time t_0. $A_X(t_0)$ defines the cross-terms amplitude.

2.2.3. Resolution Performance Measure of TFDs

Equation (3) and Figure 4 suggest that the resolution performance of a time-frequency distribution of a *two-component signal* is given by the minimum value of the difference $R = f_2 - f_1$ for which we still have a positive separation D between the components' mainlobes about their respective IFs, f_2 and f_1. For TFDs, D should ideally be as close as possible to the true difference between the actual frequencies. It is expressed as:

$$D = \frac{(f_2 - \frac{V_2}{2}) - (f_1 + \frac{V_1}{2})}{f_2 - f_1} = 1 - \frac{V_1 + V_2}{2R} \quad (4)$$

The resolution also depends on the following set of variables, all of which should be as small as possible:

a) the 1.5 dB normalised *mainlobe bandwidth* of the signal component V_k/f_k, $k = 1, 2$, which is already included in D (equation (4)),

b) the ratio of the *sidelobe magnitude* $|A_{S_k}|$ to the mainlobe magnitude $|A_{M_k}|$, $k = 1, 2$ of the components, and

c) the ratio of the *cross-term magnitude* $|A_X|$ to the mainlobe magnitude of the signal auto-terms $|A_{M_k}|$, $k = 1, 2$.

It follows that the best TFD for *multicomponent signals analysis* is the one that **minimises** the positive quantities a), b) and c), and **maximises**[2] the separation D, concurrently.

Hence, an indicator P of the resolution performance of a given TFD can be defined as [7]:

$$P = \frac{|A_S||A_X|}{|A_M|^2 D} \geq 0 \quad (5)$$

where A_M, A_S and A_X are respectively the average amplitudes of the mainlobes, sidelobes and cross-terms of any two consecutive components of the multicomponent signal, with D being their relative separation.

If $P < 0$, then there is no separation of the components, while if $P \geq 0$, P provides a measure of the resolution performance, which takes into account separation D and the effect of cross-terms (best performance is achieved by minimising P).

3. TIME-FREQUENCY SIGNAL ANALYSIS OF CLOSELY SPACED COMPONENTS USING THE B-DISTRIBUTION

3.1. Defining TFDs via Ambiguity Filtering

Different time-frequency distributions of the analytic signal $z(t)$, associated with the real signal $s(t)$, can be obtained by selecting different kernel functions $g(\nu, \tau)$ in the general expression of the quadratic class[3] [8]:

$$\rho_z(t, f) = \iiint g(\nu, \tau) e^{j2\pi\nu(t-u)} z(u + \frac{\tau}{2}) z^*(u - \frac{\tau}{2}) e^{-j2\pi f\tau} d\nu du d\tau \quad (6)$$

[2]The maximum value is $D = 1$ which is obtained when $V_1 = V_2 = 0$.

[3]All three integrals have limits from $-\infty$ to $+\infty$. Note: this formula differs from Cohen's formula by a minus sign in the first exponential.

For $g(\nu, \tau) = 1$, we obtain the Wigner-Ville distribution (WVD) of the signal [2, 9]:

$$WVD_z(t, f) = \int_{-\infty}^{\infty} z(t + \frac{\tau}{2}) z^*(t - \frac{\tau}{2}) e^{-j2\pi f\tau} d\tau \quad (7)$$

A key to understanding time-frequency relationships is through understanding of the ambiguity domain. The symmetrical ambiguity function (AF) is defined as:

$$AF_z(\nu, \tau) = \int_{-\infty}^{\infty} z(t + \frac{\tau}{2}) z^*(t - \frac{\tau}{2}) e^{-j2\pi\nu t} dt \quad (8)$$

From equations (7) and (8) we can see that the WVD and the AF are related by a two-dimensional Fourier transform [2]:

$$WVD_z(t, f)_f^t \rightleftharpoons_\tau^\nu AF_z(\nu, \tau)$$

$$WVD_z(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} AF_z(\nu, \tau) e^{-j2\pi(f\tau - \nu t)} d\nu d\tau \quad (9)$$

It was shown that a signal mapped by the AF into the Doppler-lag domain always traverses the origin of that plane, while the cross-terms, having oscillating amplitude in the time-frequency domain, are located away from the origin in the Doppler-lag plane, the distance being directly proportional to the time and frequency distance of the signal components [1].

This property of the AF has inspired researchers to look for two-dimensional kernel filters $g(\nu, \tau)$ that enhance the generalised ambiguity function, $g(\nu, \tau) AF_z(\nu, \tau)$, around its origin and suppress it elsewhere.

Using equations (6) and (9), the following expression can also be derived [2]:

$$\rho_z(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\nu, \tau) AF_z(\nu, \tau) e^{-j2\pi(f\tau - \nu t)} d\nu d\tau \quad (10)$$

Thus, quadratic TFDs may be found by filtering the symmetrical ambiguity function with $g(\nu, \tau)$ and then carrying out the two-dimensional Fourier transform. For example, for the Wigner-Ville distribution with the ambiguity domain kernel filter equal to unity, no filtering is applied to the AF, resulting in the complete preservation of the cross-terms. This in return makes the interpretation of the WVD of multicomponent signals highly difficult. The spectrogram, on the other hand, leads to a quasi-total elimination of the cross-terms to the detriment of resolution.

3.2. New Constraints for TFD Design

It was reported in [2] that for a time-frequency analysis, a TFD is expected to be real, to satisfy the marginals and to have the instantaneous frequency as its first moment with respect to frequency. These strict constraints on the kernel design in the ambiguity domain [9] led to the terminology of Cohen's class.

However, it is known that the spectrogram does not exhibit cross-terms, and does not satisfy the marginals. Yet the spectrogram is a very popular tool in practical applications, suggesting that the time and the frequency marginal constraints may not be really strictly needed in practice. What may be more important is to improve the energy concentration about the IF for monocomponent signals and improve the resolution for multicomponent signals.

Following this logic, we may therefore conclude that, to be a suitable tool for a *practical* time-frequency analysis, a TFD should verify the following minimum set of properties:[4]

1. Be real,
2. Preserve the total energy of the signal:

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho_z(t, f) dt df$$

3. Preserve the regional (component) energy: energy in the region R of the time-frequency plane bounded in time by $[t_1, t_2]$ and frequency $[f_1, f_2]$ should be:

$$E_R = \int_{f_1}^{f_2} \int_{t_1}^{t_2} \rho_z(t, f) dt df$$

4. Reduce the cross-terms, while preserving resolution by minimising measure P (defined by equation (5)),
5. Reveal the IF law of a monocomponent signal by its peak.

To satisfy these constraints, Barkat and Boashash [4, 5] recently proposed a kernel for a quadratic TFD, known as the B-distribution, defined by:

$$G(t, \tau) = \int_{-\infty}^{\infty} g(\nu, \tau) e^{j2\pi\nu t} d\nu = \left(\frac{|\tau|}{\cosh^2(t)} \right)^{\beta} \qquad (11)$$

The kernel filter $g(\nu, \tau)$ of the B-distribution (BD) was chosen in the ambiguity domain to be a two-dimensional function centred around the origin with sharp cut-off edges. In this way, the kernel would allow to retain as much auto-terms as possible while filtering out as much cross-terms. The amounts of auto-terms and cross-terms kept and filtered out are functions of the volume underneath the 2-D function $g(\nu, \tau)$. This volume can be changed by varying a single parameter β ($0 \leq \beta \leq 1$) which is application dependent.

In addition, a modification to the BD kernel (the Modified B-distribution) by authors Hussain and Boashash allows an efficient estimation of the IF laws of a multicomponent signal.

The kernel of the Modified B-distribution is defined as [10]:

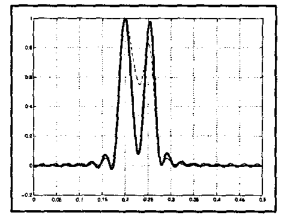
$$G(t, \tau) = \frac{\Gamma(2\alpha)}{2^{2\alpha-1}\Gamma^2(\alpha)} \frac{1}{\cosh^{2\alpha}(t)} \qquad (12)$$

where $\Gamma[\cdot]$ is the gamma function and α is a real positive number less than 1.

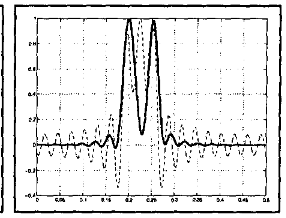
4. PERFORMANCE MEASURE AND COMPARISON OF TIME-FREQUENCY DISTRIBUTIONS

In this section, we use the newly defined measure criterion to compare the performance of the WVD, the spectrogram, the Choi-Williams distribution, the Born-Jordan distribution, Zhao-Atlas-Marks distribution, the B-distribution and the Modified B- (MB) distribution of the two-LFM-component signal defined in Section 1. For each time-frequency distribution we take a slice at the middle of the time interval and measure the parameters A_M, A_S, A_X and V. These parameters are then used to calculate the frequency separation of the components D, defined by equation (4), and the performance indicator P, defined by equation (5).

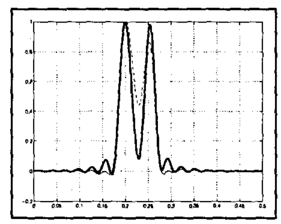
The distributions and their respective measurements parameters are recorded in Table 1, while the slices of the TFDs at the middle of the time interval are displayed in Figure 5.



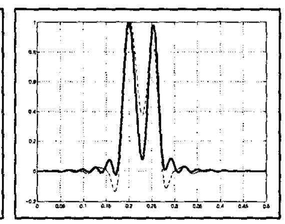
(a) BD (solid) and Spectrogram (dashed)



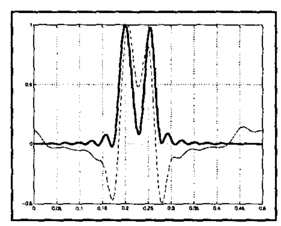
(b) BD (solid) and WVD (dashed)



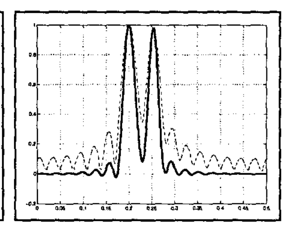
(c) BD (solid) and CWD, $\sigma = 2$, (dashed)



(d) BD (solid) and BJD (dashed)



(e) BD (solid) and ZAMD, $a = 2$, (dashed)



(f) BD (solid) and RD (dashed)

Figure 5: *Slices taken at a half of the time interval of TFDs of two closely-spaced LFMs with frequency $f_1 = 0.15 - 0.25Hz$ and $f_2 = 0.2 - 0.3Hz$. BD=B-distribution, WVD=Wigner-Ville distribution, CWD=Choi-Williams distribution, BJD=Born-Jordan distribution, ZAMD=Zhao-Atlas-Marks Distribution, and RD=Rihaczek Distribution*

The TFD which gives the smallest positive P is the TFD with the best performance when used to analyse multicomponent signals. In our case, the B-distribution ($\beta = 0.01$) yields the smallest value for P ($P = 1.04 \times 10^{-2}$) and hence is regarded as best. Similar results were obtained with other types of signals.

[4] Note that the selection of a complete set of properties would be application dependent.

TFD	A_M	A_S	A_X	$V[Hz]$	D	*Performance Measure P*
B-Distribution (BD), $\beta = 0.01$	0.9890	0.0796	0.0810	0.0197	0.6337	1.04×10^{-2}
Modified B-Distribution (MBD), $\alpha = 0.01$	0.9885	0.0861	0.0947	0.0199	0.6298	1.32×10^{-2}
Born-Jordan Distribution (BJD)	0.9320	0.1227	0.3798	0.0236	0.5164	1.04×10^{-1}
Choi-Williams Distribution (CWD), $\sigma = 2$	0.9335	0.0211	0.4415	0.0258	0.4766	2.25×10^{-2}
Spectrogram (Hanning window)	0.9119	0.0493	0.5527	0.0323	0.3557	9.21×10^{-2}
Rihaczek Distribution (RD)	0.9823	0.2945	0.3446	0.0289	0.4630	2.71×10^{-1}
Wigner-Ville Distribution (WVD)	0.9153	0.4134	1	0.0140	0.7558	6.53×10^{-1}
Zhao-Atlas-Marks Distribution (ZAMD), $a = 2$	0.9146	0.4822	0.4796	0.0238	0.4331	6.08×10^{-1}

Table 1: *Measurements parameters and the performance indicator P of TFDs (slices taken at the half of the signal time interval) of two closely-spaced LFMs with frequency* $f_1 = 0.15 - 0.25Hz$ *and* $f_2 = 0.2 - 0.3Hz$

4.1. Optimisation of the B-Distribution Parameter β Using the Performance Measure P

The performance measure P can be used to optimise the value of the smoothing parameters of a given TFD. *One approach* would be to take consecutive slices of the TFD, find measure P for each of the slices, and average all such obtained measures for a given value of the TFD parameter to obtain the average performance measure P_{av}. Repeating this procedure over a range of values of the smoothing parameter, it is possible, by identifying the one which results into smallest P_{av}, to obtain the *optimal value* of the smoothing parameter of the TFD considered.

For example, using the measure P, we can optimise the parameter β of the B-distribution for the signal in Section 1. Simulations have shown that $\beta = 0.01$ gives visually most appealing results for various multicomponent signals [4]. However, this value can be refined by applying the above described optimisation procedure.

By calculating P_{av} for $\beta \in [0, 1]$ with the increment of 10^{-5} and for the distribution slices 16:112 (note that the signal length is $N = 128$)[5] we find the optimal value of the smoothing parameter of the B-distribution to be $\beta_{opt} = 9.9 \times 10^{-4}$ ($P_{av} = 9.1 \times 10^{-3}$). Indeed, a reduction in P_{av} value of approximately 2×10^{-3} is achieved if the smoothing parameter of the B-distribution is optimised, when compared to $P_{av} = 1.1 \times 10^{-2}$ of the B-distribution with $\beta = 0.01$.

5. CONCLUSION

This paper has presented two key results which we believe to be fundamental to a better understanding and use of time-frequency signal analysis tools.

The first key result is a definition of an objective criterion to compare the resolution performance of time-frequency distributions for multicomponent signals analysis using a quantitative measure of goodness for TFDs. This result fills an obvious need in that until now the comparison of the resolution performance of TFDs was primarily based on a visual impression of the plots of TFDs.

The second key result is an improvement in the design of tools for high resolution time-frequency analysis of multicomponent signals. By removing limitations in the way desirable properties of quadratic TFDs were previously chosen, a new set of design criteria has been defined. It was found that such defined B-distribution outperforms other existing distributions in terms of time-frequency resolution, as well as cross-terms suppression, when used to represent signals with closely-spaced components in the time-frequency domain.

The combination of these two results is an important breakthrough for the field of time-frequency signal analysis. It opens the way for further research in developing high resolution DSP tools for non-stationary (time-varying) signals by removing unnecessary limitations, and providing a measure of quality of TFDs.

[5] We avoid calculations of the measure P for the first and the last eighth of the TFD slices (i.e. the beginning and the end of the TFD in time) since it is known [2] that in these regions of the time-frequency plane the components resolution is always significantly degraded.

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